Gardiner

Review of basic Quantum Mechanics and Optics

Harmonic oscillator states

$$\hat{q} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^+) \qquad \hat{p} = -i\sqrt{\frac{\hbar\omega m}{2}}(\hat{a}-\hat{a}^+)$$

Position and momentum operators

$$[\hat{q},\hat{p}] = i\hbar \qquad (\text{and } [\hat{a},\hat{a}^+] = 1) \text{ or } \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$$

Consider the variance of the coherent state $|\alpha\rangle = e^{-1/2|\alpha|^2} e^{\alpha a^+}|0\rangle$

$$\langle\Delta\hat{q}^2\rangle \hat{=} \langle(\hat{q}-\langle\hat{q}\rangle)^2\rangle = \langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2$$

for coherent state we have:

$$\langle\alpha|\Delta\hat{q}^2|\alpha\rangle = \langle\alpha|\hat{q}^2|\alpha\rangle - (\langle\alpha|\hat{q}|\alpha\rangle)^2$$

$$= \frac{\hbar}{2m\omega}\langle\alpha|\hat{a}^2 + \underbrace{\hat{a}\hat{a}^+}_{\hat{a}^+\hat{a}+1} + \hat{a}^+\hat{a} + (a^+)^2|\alpha\rangle - \langle\hat{q}\rangle^2$$

$$= \frac{\hbar}{2m\omega}\cdot\langle\alpha|(\alpha^2 + \alpha\alpha^* +)1 + \alpha^*\alpha + \alpha^{*2})|\alpha\rangle - \langle\hat{q}\rangle^2$$

$$= \frac{\hbar}{2m\omega}\cdot(1 + (\alpha+\alpha^*)^2 - \langle\hat{q}\rangle^2)$$

$$= \frac{\hbar}{2m\omega}$$

analogous

$$\langle\alpha|\Delta\hat{p}^2|\alpha\rangle = \frac{\hbar m\omega}{2} \Rightarrow \boxed{\langle\Delta\hat{p}^2\rangle\langle\Delta\hat{q}^2\rangle = \left(\frac{1}{2}\hbar\right)^2 = \frac{\hbar^2}{4}}$$

for a coherent state (i.e a MINIMUM UNCERTAINTY STATE).

Compare this to the VACUUM STATE: $|0\rangle$.

$$\langle 0|\Delta\hat{q}^2|0\rangle = \langle 0|\hat{q}^2|0\rangle = \frac{\hbar}{2m\omega}\cdot\langle 0|\hat{a}^{+2} + \hat{a}^+\hat{a} + \underbrace{\hat{a}\hat{a}^+}_{a^+a+1} + \hat{a}^2|0\rangle = \frac{\hbar}{2m\omega}$$

i.e $|0\rangle$ is also an minimum uncertainty state

$$\langle\Delta\hat{p}^2\rangle_0\langle\Delta\hat{q}^2\rangle_0 = \frac{\hbar^2}{4} \quad \text{for } |0\rangle.$$

Note: not true for all states (e.g. thermal states are not minimum uncertainty states.)

The Quantum Optical Master Equation / Heisenberg Langevin Equation

Problem Formulation: Consider Harmonic oscillator $\hat{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$ (di

Equation of Motion: $\dot{q} = p/m = \frac{\partial H}{\partial p}$; $\dot{p} = -m\omega^2 q = -\frac{\partial H}{\partial q}$

To add damping we simply add term: $-\gamma p$ i.e. $\ddot{q} + \gamma \dot{q} + \omega^2 q = 0$

Lets consider a quantum mechanical system

$$[\hat{q},\hat{p}] = i\hbar \qquad \text{and: } i\hbar \dot{\hat{q}} = -i\hat{p}/m$$
$$i\hbar \dot{\hat{p}} = -m\omega^2 \hat{q} - \gamma \hat{p}$$

Damping added

$a \equiv \frac{1}{\sqrt{2\hbar\omega m}}(m\omega\hat{q} + i\hat{p}$

$a^\dagger \equiv \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{q} - i$

$\hat{H} \hat{=} \hbar\omega a^\dagger a$

consequence $\frac{d}{dt}[\hat{q},\hat{p}] = \frac{d}{dt}[(\hat{q}\hat{p}) - \hat{p}\hat{q}]$
$$= \dot{\hat{q}}\hat{p} + \hat{q}\dot{\hat{p}} - \dot{\hat{p}}\hat{q} - \hat{p}\dot{\hat{q}} = -\gamma[\hat{q},\hat{p}]$$

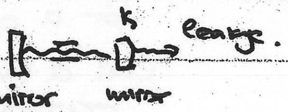
Hence: $[\hat{q}(t),\hat{p}(t)] = i\hbar e^{-\gamma t}$

Thus the commutator decays with time; a severe problem!

This violates the Heisenberg uncertainty principle which now becomes

$$\sqrt{\langle\Delta\hat{q}^2\rangle\langle\Delta\hat{p}^2\rangle} \geq \frac{1}{2}\hbar\,(e^{-\gamma t}) \to 0 \text{ for } t \to \infty$$

In which examples does this dissipation play a role? Eg in a

laser system:  OPEN QUANTUM SYSTEM. (Caldeira and Leggett)

lossy LC circuit oscillator 

SYSTEM and RESERVOIR APPROACH: Wigner-Weisskopf: Heisenberg-Langevin Approach

A simple example: $H = \underbrace{\hbar\omega\hat{a}^\dagger\hat{a}}_{\text{SYSTEM}} + \underbrace{\sum \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k}_{\text{BATH}}$ and: $a^\dagger a|n\rangle = n|n\rangle$

SYSTEM-BATH COUPLING

$$\boxed{\hat{H}_{int} = \sum_k g_k(\hat{a}^\dagger\hat{b}_k + \hat{a}\hat{b}_k^\dagger)}$$ Interaction Hamiltonian

or classically:

$$\hat{H} = \tilde{g}\cdot\hat{q}_1\hat{p}_2 \propto g\,(a+a^\dagger)(b+b^\dagger)$$ classically

References

System plus reservoir approach

Scully "Quantum optics"

$$\dot{\hat{a}} = \frac{-i}{i\hbar}[\hat{H},\hat{a}] = -i\omega\,\hat{a}(t) - i\sum_k g_k \hat{b}_k(t)$$

$$\dot{\hat{b}}_k = -i\omega_k \hat{b}_k - i g_k \hat{a}(t)$$

Formally integrate out the equation for b_k:

$$\hat{b}_k(t) = \hat{b}_k(0)e^{-i\omega_k t} - i g_k \int_0^t dt'\,\hat{a}(t')e^{-i\omega_k(t-t')}$$

insert this back into equation for a:

$$\dot{\hat{a}}(t) = -i\omega\hat{a} - \sum_k g_k^2 \int_0^t dt'\,\hat{a}(t')e^{-i\omega_k(t-t')} + \hat{f}_a(t)$$

$$\hat{f}_a(t) = -i\sum_k g_k \hat{b}_k(0)e^{-i\omega_k t}$$

Noise Operator (depends on reservoir only)

Transform to slowly varying envelope

$$\tilde{a}(t) = \hat{a}(t)e^{+i\omega t}$$

Thus:

$$\dot{\tilde{a}} = -\sum_k g_k^2 \int_0^t dt'\,\hat{a}(t')e^{-i[\omega_k-\omega](t-t')} + \hat{F}_{\tilde{a}}(t)$$

$$\hat{F}_{\tilde{a}}(t) = e^{+i\omega t}\hat{f}_a(t) = -i\sum g_k \hat{b}_k(0)e^{-i(\omega_k-\omega)t}$$

Simplify expressions:

$$\sum_k g_k^2 \int_0^t \hat{\tilde{a}}(t')e^{-i(\omega_k-\omega)(t-t')} \approx \frac{1}{2}\kappa\,\tilde{a}(t)$$

DECAY

using $\int_{-\infty}^{\infty} e^{i(\omega+\omega_k)(t-t')} = 2\pi\delta(t-t')$

* proof on next page

$$\boxed{\kappa = 2\pi|g(\omega)|^2 \underbrace{D(\omega)}_{\text{Density of states}}}$$

1st Markov approximation: coupling is frequency independent → pull $g(\omega)$ out of integral over ω.

Hence we have:

$$\frac{d}{dt}\hat{\tilde{a}}(t) = -\frac{\kappa}{2}\hat{\tilde{a}}(t) + \underbrace{\hat{F}_{\tilde{a}}(t)}_{\text{Noise operator}}$$

or in original frame

$$\boxed{\frac{d}{dt}\hat{a}(t) = -i\omega\hat{a}(t) - \kappa/2\,\hat{a}(t) + \hat{f}_a(t)}$$

forward Q.L. Eq

QUANTUM LANGEVIN EQUATION

Quantum Langevin Equations

<u>Addendum</u> simplification step used in previous page:

$$\sum_k g_k^2 \int_0^t \hat{a}(t') e^{-i(\omega_k-\omega)(t-t')} dt'$$

Note
$D(\omega)d\omega$: # modes in $\omega \ldots \omega + d\omega$

Free space EM modes $D(\omega) = \frac{\omega^2}{\pi^2 c^3}$

Convert summation to an integral

$$\sum_k g_k^2 = \int_0^\infty d\omega_k D(\omega_k) |g(\omega_k)|^2 \quad (*)$$ where $D(\omega)$ is the density of states (D.O.S)

<u>1st Markov approximation</u>: the coupling g_k is approximately (around system frequency) frequency independant. Using (*) yields

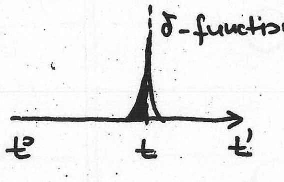
$$\int_0^\infty d\omega_k D(\omega_k) |g(\omega_k)|^2 \int_0^t \hat{\tilde{a}}(t') e^{-i(\omega_k-\omega)(t-t')} dt'$$

$$\cong D(\omega)|g(\omega)|^2 \int_0^t \hat{\tilde{a}}(t') \underbrace{\int_0^\infty d\omega\, e^{-i(\omega_k-\omega)(t-t')}}_{2\pi\delta(t-t')} dt'$$

1st Markov approx. $\omega_k \to \omega$

$$= D(\omega)|g(\omega)|^2 \cdot \int_0^t \tilde{a}(t') 2\pi\delta(t-t') dt'$$

<u>Note:</u> $$\int_0^t \tilde{a}(t') 2\pi\delta(t-t') dt' = 2\pi\, \tilde{a}(t) \cdot \underbrace{\frac{1}{2}}_{\text{extra factor } 1/2.}$$



since: $$\int_0^\infty 2\pi \tilde{a}(t)\delta(t-t')dt' = \tilde{a}(t) = \int_0^t 2\pi\tilde{a}(t)\delta(t-t')dt' + \int_t^\infty 2\pi\tilde{a}(t)\delta(t-t')dt'$$

(both contributions are equal)

Thus:

$$= 2\pi D(\omega)|g(\omega)|^2 \frac{1}{2} \tilde{a}(t)$$

$$\equiv \kappa\, \tilde{a}(t) \cdot \frac{1}{2}$$

$$\boxed{\kappa = 2\pi D(\omega)|g(\omega)|^2}$$

Damping rate

Hence

$$\boxed{\frac{d}{dt}\hat{\tilde{a}}(t) = -\frac{\kappa}{2}\hat{\tilde{a}}(t) + \hat{F}_a(t)}$$

Quantum Langevin Eqs.

Quantum Theory of Damping
System plus reservoir approach

$$[\hat{\tilde{a}}(t), \hat{\tilde{a}}^+(t)] = e^{-\kappa t} \quad \text{at time } t=0$$

$[\hat{F}(t), \hat{F}^+(t)]$ helps to maintain the commutator, This is a Manifestation of the Fluctuation Dissipation Theorem.

Assumptions on bath modes: (Thermal States)

$$\boxed{\begin{aligned} &\langle \hat{b}_k(0)\rangle = \langle \hat{b}_k^+(0)\rangle = 0 \\ &\langle \hat{b}_k^+(0)\hat{b}_{k'}(0)\rangle = \bar{n}_k \delta_{kk'} \\ &\langle \hat{b}_k(0)\hat{b}_{k'}^+(0)\rangle = (\bar{n}_k+1)\delta_{kk'} \\ &\langle \hat{b}^+(0)\hat{b}^+(0)\rangle = \langle \hat{b}(0)\hat{b}(0)\rangle = 0 \end{aligned}}$$

$\bar{n}_k$: mean thermal occupa

* ρ [density matrix] has only entries on diagonal. for thermal state.

let us calculate: $\langle \hat{F}(t)\rangle = 0$ and $\langle \hat{F}^+(t)\rangle = 0$

$$\langle \hat{F}^+(t)\hat{F}(t)\rangle_R = \sum_k\sum_{k'} g_k g_{k'} \langle \hat{b}_k^+ \hat{b}_{k'}\rangle \exp[-i(\omega_k-\omega)t - i(\omega_{k'}-\omega)t']$$

$$\overset{*}{=} \sum_k g_k^2 \bar{n}_k \exp(i[\omega_k-\omega](t-t'))$$

$$= \int_0^\infty D(\omega_k)\cdot|g(\omega_k)|^2 d\omega_k\, e^{i(\omega_k-\omega)(t-t')}\, \bar{n}(\omega_k)$$

Note that we can assume that $D(\omega_k)$ = const over range where integral contributes and thus (1st Markov approximation)

$$\langle \hat{F}^+(t)\hat{F}(t)\rangle = \underbrace{\bar{n}(\omega)}\, D(\omega_k)|g(\omega)|^2 \int_0^\infty d\omega_k\, e^{-i[\omega-\omega_k](t-t')}$$

$$= D(\omega)|g(\omega)|^2\,\delta(t-t')\cdot\bar{n}(\omega) \quad \wedge \quad 2\pi|g(\omega)|^2 D(\omega) = \kappa$$

$$\boxed{\langle \hat{F}^+(t)\hat{F}(t')\rangle \mathrel{\hat{=}} \kappa\,\bar{n}(\omega)\,\delta(t-t')}$$

This is in perfect analogy with the classical Langevin Equations

similarly:

$$\boxed{\begin{aligned} &\langle \hat{F}(t)\hat{F}^+(t')\rangle = \kappa[\bar{n}(\omega)+1]\delta(t-t') \\ &\langle \hat{F}(t)\hat{F}(t')\rangle = 0 \end{aligned}}$$

Let us look at the quantity: $\langle \hat{F}^+(t)\hat{a}(t)\rangle$ (we will need this later)

$$\langle \hat{F}^+(t)\hat{a}(t)\rangle = \langle \hat{F}^+(t)[\hat{a}(0)e^{-\kappa/2\, t} + \int^t dt'\, e^{-\kappa/2(t-t')}\hat{F}(t')]$$

$$= \underbrace{\langle \hat{F}^+(t)\hat{a}(0)\rangle}_{\text{statistically independent}} + \int_0^t \underbrace{\langle \hat{F}^+(t)\hat{F}(t')\rangle}_{\delta(t-t')} dt'\, e^{-\kappa/2(t-t')} = \bar{n}(\omega)\cdot\frac{\kappa}{2}$$

IMPORTANT!

Quantum Theory of Damping

Similarly $\boxed{\langle a^{+}(t) \hat{F}(t) \rangle = \kappa/2\, \bar{n}(\omega)}$

Applying the results:

$$d_t \langle \hat{\tilde{a}}(t) \rangle_R = -\frac{1}{2}\kappa \langle \hat{\tilde{a}}(t) \rangle_R + \underbrace{\langle \hat{F}(t') \rangle_R}_{=0}$$

Thus $\boxed{\langle \hat{\tilde{a}}(t) \rangle = \langle \hat{\tilde{a}}(0) \rangle e^{-\kappa/2\, t}}$ [Equation for Amplitude] system decays.

Next:

$$d_t \langle \tilde{a}^{+}(t) \tilde{a}(t) \rangle = d_t \langle \hat{n}(t) \rangle = \langle \frac{d}{dt} \tilde{a}^{+}(t) \tilde{a}(t) \rangle_R + \langle \tilde{a}^{+}(t) \frac{d}{dt} \tilde{a}(t) \rangle_R$$

$$= -2 * \frac{\kappa}{2} \langle \hat{\tilde{a}}^{+}(t) \hat{a}(t) \rangle_R + \underbrace{\langle \tilde{F}(t) \tilde{a}(t) \rangle_R + \langle \hat{\tilde{a}}(t) \hat{F}(t) \rangle_R}_{\text{calculated before}} + \kappa$$

$$= -\kappa \langle \tilde{a}^{+}(t) \tilde{a}(t) \rangle_R + \kappa \bar{n}(\omega) + \kappa$$ [Equation for occupancy / Energy]

$$\Leftrightarrow \boxed{d_t \langle \tilde{a}(t) \tilde{a}^{+}(t) \rangle_R = -\kappa \langle \hat{\tilde{a}}(t) \tilde{a}^{+}(t) \rangle_R + \kappa(\bar{n}(\omega) + 1)}$$

Important: Energy decays faster than amplitude (x2)

and we see now that the COMMUTATOR

$$\frac{d}{dt} [\tilde{a}(t), \tilde{a}^{+}(t)] = \frac{d}{dt}(\tilde{a}(t) \tilde{a}^{+}(t)) - \frac{d}{dt}(a^{+} a)$$

$$= -\kappa [\tilde{a}(t), a^{+}(t)] + \kappa$$

Steady state $\boxed{[\tilde{a}(t), a^{+}(t)] = 1}$ Theory has preserved commutator at all times.

A further strength of the approach is to calculate using the Heisenberg Langevin Equation the SPECTRUM.

$$\dot{\tilde{a}}(t) = -\frac{\kappa}{2} \hat{\tilde{a}}(t) + \hat{F}(t)$$

thus $$\hat{a}(t_i + T) = \tilde{a}(t) \exp(-T\kappa/2) + \int_t^{t+T} dt' e^{-(t_i + T - t')} \hat{F}_a(t)$$

$$\Leftrightarrow \langle \tilde{a}^{+}(t_i) \tilde{a}(t_i + T) \rangle_R = \langle \tilde{a}^{+}(t_i) \tilde{a}(t_i) \rangle \exp(-T\kappa/2)$$

$$\Leftrightarrow \langle a^{+}(t_i) a(t_i + T) \rangle_R = \langle a^{+}(t_i) a(t_i) \rangle \exp(-T\kappa/2 - i\omega_c T)$$ old Basis

Fourier Transform:

$$S_{aa}(\omega) = \frac{1}{\pi} \mathrm{Re} \int_0^{\infty} \langle a^{+}(t) a(t+T) \rangle_R e^{i\omega T} dT$$

$$\boxed{S_{aa}(\omega) = \frac{\langle n \rangle}{\pi} \frac{\kappa/2}{(\omega - \omega_c) + \kappa^2/4}}$$

Lorentzian Spectrum *
The NORMALLY ORDERED SPECTRUM.

Non Equilibrium Stat Mech

Lecture 10 : OPEN QUANTUM SYSTEMS

Dissipation in Quantum Mechanics

overview: derivation of a quantum Markov process leads Quantum Langevin eqs and the Quantum Optical Master Equation.

Brief Review: Quantum Mechanics of Harmonic Oscillator (see: STOCHASTIC METHODS C. GARDINER)

creation and annihilation (destruction operator)

$$\boxed{[\hat{a},\hat{a}^+] = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1}$$ commutation relation

Eigenstates $\boxed{\hat{a}^+\hat{a}\,|n\rangle = n|n\rangle}$ $n \in \mathbb{N}_0$

(for proof see Cohen Tannoudji: Quantum Mechanics, Part I)

and: $$\boxed{\begin{aligned} a|n\rangle &= \sqrt{n}\,|n-1\rangle \\ a^+|n+1\rangle &= \sqrt{n+1}\,|n+1\rangle \end{aligned}}$$

Orthogonality

$$\langle n|m\rangle = \delta_{m,n}$$

Number Operator

$$\boxed{\hat{N} = \hat{n} = \hat{a}^+\hat{a}}$$

Harmonic Oscillator: $\hat{H} = (a^+a + \frac{1}{2})\hbar\omega$ $\hbar$: Planck constant

$\frac{\hbar\omega}{2}$: Zero Point Energy

$$\boxed{\begin{aligned} &\hat{H}|n\rangle = E_n|n\rangle \text{ and Eigenenergy} \\ &E_n = (\hbar\omega)(n+\tfrac{1}{2}) \end{aligned}}$$

Schrödinger Equation

Physical state $|A,t\rangle$

$$H|A,t\rangle = i\hbar\partial_t|A,t\rangle \quad (\text{or: } \hat{H}\psi(t) = -i\hbar\partial_t\psi(t))$$

One can expand state

$$H|A,t\rangle = \underbrace{\sum_n |n\rangle\langle n|}_{\mathbb{1}} A,t\rangle$$

$$\Rightarrow i\hbar\partial_t|A,t\rangle = i\hbar\left(\sum_n |n\rangle\partial_t\langle n|A,t\rangle\right) = \sum_n H|n\rangle\langle n|A,t\rangle$$

$$= \sum_n (n+\tfrac{1}{2})\hbar\omega|n\rangle\langle n|A,t\rangle$$

$\Rightarrow \langle n|A,t\rangle = e^{-iE_n t/\hbar} \langle n|A,0\rangle$ Evolution of system

e.g. $|A,t=0\rangle = |1\rangle \rightarrow |A,t\rangle = |1\rangle e^{-i\omega t}$; $|A,t=0\rangle = |3\rangle = |3\rangle e^{-i3\omega t}$

COHERENT STATES (no proof)

Definition (Roy Glauber)
$$\boxed{|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2) \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

Scalar product $\langle\alpha|\beta\rangle = \exp(\alpha^*\beta - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2)$

$$|\langle\alpha|\beta\rangle|^2 = \exp(-|\alpha-\beta|^2)$$

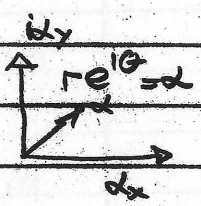
Thus coherent states of different eigenvalue are not orthorgonal. They form an overcomplete basis.

Completeness Formula:

$$\boxed{\mathbb{1} = \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \, |\alpha\rangle\langle\alpha|}$$ $\mathbb{C}$: complex plane

Proof: assume general state $|A\rangle = \sum_n A_n |n\rangle$

Note $\alpha = \alpha_x + i\alpha_y$



$\Rightarrow \frac{1}{\pi} \int d\alpha^2 (|\alpha\rangle\langle\alpha|)|A\rangle = \int \frac{1}{\pi} \sum_n d^2\alpha A_n |\alpha\rangle\langle\alpha|n\rangle$

substitute polar coordinates $\alpha = re^{i\theta}$ $d^2\alpha = r\,dr\,d\theta$

Thus:
$$\frac{1}{\pi} \sum_{m,n} \int_0^\infty\int_0^{2\pi} A_n e^{-r^2} r^{n+m} e^{i(m-n)\theta} (n!m!)^{-1/2} |m\rangle r\,dr\,d\theta$$
$$= 2 \sum_n \int_0^\infty A_n e^{-r^2} r^{2n+1} (n!)^{-1} |n\rangle dr$$

where we used: $2\delta_{n,m} = \frac{1}{\pi}\int_0^{2\pi} e^{i(m-n)\theta} d\theta$

Thus: and using
$$\int_0^\infty dr\, e^{-r^2} (r^{2n+1}) = n!/2$$

$\Rightarrow \quad = 2\cdot(n!)^{-1}\left[\frac{n!}{2}\right] = 1$ Thus identity $\mathbb{1}$ q.e.d.

side Note $\boxed{||\alpha\rangle = \exp(\tfrac{1}{2}|\alpha|^2)|\alpha\rangle}$ are called BARGMANN STATES

Expansion of an operator in coherent states ($\hat{T}$)

$$\hat{T} = \mathbb{1}\hat{T}\mathbb{1} = \frac{1}{\pi^2}\iint d^2\alpha\, d^2\beta\, |\alpha\rangle \underbrace{\langle\alpha|\hat{T}|\beta\rangle}_{T^*(\alpha^*,\beta)} \langle\beta| \qquad \partial^2\alpha \hat{=} \mathrm{Re}\{\partial\alpha\}\,\mathrm{Im}\{\partial\alpha\}$$

$$= \frac{1}{\pi^2}\iint d^2\alpha\, d^2\beta\, |\alpha\rangle\langle\beta|\, T(\alpha^*,\beta)\exp\left(-\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2\right)$$

where $*$ $T(\alpha^*,\beta) = \exp\left(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2\right)\langle\alpha|T|\beta\rangle = \langle\alpha\|T\|\beta\rangle$

For example operator $\boxed{T = (\hat{a}^+)^m(\hat{a})^n}$

then: $T(\alpha^*,\beta) = \langle\alpha|(a^+)^m(\hat{a})^n|\beta\rangle \exp\left(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2\right)$

$= (\alpha^*)^m \beta^n \langle\alpha|\beta\rangle \exp\left(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2\right)$

(note that $\langle\alpha|a^+ = \alpha^*\langle\alpha|$)

$$\boxed{T(\alpha^*,\beta) = (\alpha^*)^m(\beta)^n \exp(\alpha^*\beta)}$$

Use this to calculate its Expectation value:

$$\langle\alpha|T|\alpha\rangle \underset{*}{=} \sum_{n,m}\langle n|T|m\rangle \cdot e^{|\alpha|^2}(\alpha^*)^n(\alpha)^m \frac{1}{\sqrt{n!}\sqrt{m!}}$$

(replace $|\alpha\rangle$ by Fock state expansion)

so that (rewrite using derivatives r.h.s) $\langle n|T|m\rangle = \sqrt{n!}\sqrt{m!}\,\dfrac{\partial^n}{\partial\alpha^{*n}}\dfrac{\partial^m}{\partial\alpha^m}\left(e^{\alpha\alpha^*}\langle\alpha|T|\alpha\rangle\right)$

Formal derivatives in ANALYTIC FUNCTION THEORY
$$\begin{cases} \alpha = x + iy \\ \dfrac{\partial}{\partial\alpha} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right) \\ \dfrac{\partial}{\partial\alpha^*} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right) \end{cases}$$

COHERENT STATES

$$\boxed{a|\alpha\rangle = \alpha|\alpha\rangle \quad \text{and} \quad \langle\alpha|a^+ = \alpha^*\langle\alpha|}$$

NORMALLY ORDERED OPERATORS

$\langle\alpha|a^+aa^+|\beta\rangle$ is not normally ordered

$\langle\alpha|a^+aa^+|\beta\rangle = \langle\alpha|\, a^+(a^+a+1)\,|\beta\rangle \qquad [a,a^+] = 1$

$= \alpha\langle\alpha|\underbrace{a^+a^+a}_{\text{Normally ordered}} + a^+|\beta\rangle \qquad aa^+ - a^+a = 1$

$= (\alpha^{*2}\beta + \alpha^*)\langle\alpha|\beta\rangle$

":" Normal ordering (important in e.g. optical photodetection)

$$:(a+a^{\dagger})(a+a^{\dagger}): = a^{\dagger 2} + a^2 + 2a^{\dagger}a$$

(i.e. destruction operators to right of creation operators)

Poisson number distribution of coherent state

$|n\rangle$: number state

$$\boxed{P_{\alpha}(n) = |\langle n|\alpha\rangle|^2 = \left|\exp(-\tfrac{1}{2}|\alpha|^2)\frac{\alpha^n}{\sqrt{n!}}\right|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}}$$

Poisson distribution with a mean $|\alpha|^2$.

$$\langle\hat{N}\rangle = \langle\alpha|\hat{N}|\alpha\rangle = \sum_n nP(n) = \ldots = |\alpha|^2$$

$$\langle\hat{N}^2\rangle = \langle\alpha|a^{\dagger}a\,a^{\dagger}a|\alpha\rangle$$

$$= \langle\alpha|a^{\dagger}(a^{\dagger}a+1)a|\alpha\rangle = \langle\alpha|a^{\dagger}a^{\dagger}aa + a^{\dagger}a|\alpha\rangle$$

$$= |\alpha|^4 + |\alpha|^2$$

Thus: $\langle\hat{N}(\hat{N}-1)\rangle = |\alpha|^4 = \langle\hat{N}\rangle^2$ i.e. $\boxed{\langle\hat{N}^2\rangle - \langle\hat{N}\rangle = \langle\hat{N}\rangle^2}$

PROPERTY OF A POISSON PROCESS

~~REVIEW~~

DENSITY MATRIX AND PROBABILITIES

$\langle\hat{M}\rangle = \langle\psi|\hat{M}|\psi\rangle$ expectation value of operator $\hat{M}$

~~For a non pure state~~

$$\langle\hat{M}\rangle = \sum_a P(a)\langle\psi_a|\hat{M}|\psi_a\rangle$$

Definition of the density Matrix $\boxed{\rho \equiv \sum_a P(a)|\psi_a\rangle\langle\psi_a|}$

Then: $\boxed{\langle M\rangle = \mathrm{Tr}\{\rho M\} \quad \text{where } \mathrm{Tr}\{B\} \equiv \sum_n \langle n|B|n\rangle}$

Thus since: $\mathrm{Tr}\{\rho M\} = \sum_{n,a} P(a)\langle n|\psi_a\rangle\langle\psi_a|\hat{M}|n\rangle$

$$= \sum_{n,a} P(a)\langle\psi_a|M\underbrace{|n\rangle\langle n|\psi_a\rangle}_{\sum |\psi_a\rangle}$$ since: $|\psi_a\rangle = \sum_n \langle n|\psi_a\rangle|n\rangle$

$$= \sum_a P(a)\langle\psi_a|M|\psi_a\rangle$$

$$= \langle M\rangle$$

$\mathrm{Tr}\{\rho M\} = \mathrm{Tr}\{M\rho\}$ cyclic property of the trace

Properties of the density Matrix

i) $\mathrm{Tr}\{\rho\} = 1 = \sum_a p_a \cancel{\langle\psi_a|\psi_a\rangle} = \sum p_a = 1$ (sum of diagonal elements "Populations")

(ii) ρ is semi definative for state $|A\rangle$

$\langle A|\rho|A\rangle = \sum p_a |\langle A|\psi_a\rangle|^2 \geq 0$

(iii) For a PURE state $\rho^2 = \rho$ and thus conversely $p_a = \delta_{a,a_0}$

$\left(\rho^2 = \sum_i p_a p_b |\psi_a\rangle\langle\psi_b|\langle\psi_a|\psi_b|\right)$

(iv) $\mathrm{Tr}\{\rho^2\} \leq \mathrm{Tr}\{\rho\}$

Equation of Motion for ρ: VON NEUMANN'S EQUATION

Schrödinger Equation: $\boxed{\hat{H}|\psi\rangle = -i\hbar\partial_t|\psi\rangle}$ $\qquad \rho = \sum_a p_a |\psi_a\rangle\langle\psi_a|$

$$\partial_t\rho = \sum p_a [\underbrace{\partial_t|\psi_a\rangle}\langle\psi_a| + |\psi_a\rangle\underbrace{\langle\partial_t\psi|}]$$

$$= \frac{1}{i\hbar} \quad i\hbar\partial_t|\psi_a\rangle = H|\psi_a\rangle \qquad \Rightarrow -i\hbar\partial_t\langle\psi| = \langle\psi|H$$

$$\partial_t\rho = \frac{\sum p_a}{i\hbar}\left(\hat{H}|\psi_a\rangle\langle\psi_a| - |\psi_a\rangle\langle\psi_a|\hat{H}\right)$$

$$\boxed{-i\hbar\,\partial_t\rho = \frac{\cdot}{i\hbar}(H\rho - \rho H) = \frac{[H,\rho]}{i\hbar}}$$ NEUMANN EQUATION ("QUANTUM LIOUVILLE THEOREM")

$$\boxed{\rho(t) = \exp\left(-\frac{i\hat{H}t}{\hbar}\right)\rho(0)\exp\left(\frac{i\hat{H}t}{\hbar}\right)}$$

Glauber P representation (Sudarshan Phys Rev Lett 1963)

assume that $\boxed{\rho = \int d^2\alpha \,\underbrace{P(\alpha,\alpha^*)}_{\text{"quasi probability density"}}|\alpha\rangle\langle\alpha|}$

Note that: $\int d^2\alpha\, P(\alpha,\alpha^*) = 1$

since $1 = \langle\alpha|\alpha\rangle = \mathrm{Tr}\{\rho\} = \int d^2\alpha\, P(\alpha,\alpha^*) \underbrace{\sum_n \langle n|\alpha\rangle\langle\alpha|n\rangle}_{=1}$

OPERATOR CORRESPONDENCE:

$a|\alpha\rangle = \alpha|\alpha\rangle$ $\boxed{a^+||\alpha\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}}\sqrt{n+1}\,|n+1\rangle = \frac{\partial}{\partial\alpha}||\alpha\rangle}$

and: $\langle\alpha||a = \frac{\partial}{\partial\alpha^*}\langle\alpha||$ $\left(\langle\alpha|a^+ = \langle\alpha|\alpha^*\right)$

operator correspondences

$$a^+\rho = a^+\int d^2\alpha\, ||\alpha\rangle\langle\alpha|| e^{-\alpha\alpha^*} P(\alpha,\alpha^*)$$

$$= a\int d^2\alpha\, \frac{\partial}{\partial\alpha}||\alpha\rangle\langle\alpha|| e^{-\alpha\alpha^*} P(\alpha,\alpha^*)$$

$$= \int d^2\alpha\, ||\alpha\rangle\langle\alpha|| e^{-\alpha\alpha^*}\left(\alpha^* - \frac{\partial}{\partial\alpha}\right)P(\alpha,\alpha^*) \quad \text{I. By parts.}$$

$a\rho \to \alpha P(\alpha,\alpha^*)$	(This will be needed later to solve the Quantum Master Equations)
$a^+\rho \to \left(\alpha^* - \frac{\partial}{\partial\alpha}\right) P(\alpha,\alpha^*)$	
$\rho a \to \left(\alpha - \frac{\partial}{\partial\alpha^*}\right) P(\alpha,\alpha^*)$	
$\rho a^+ \to (\alpha^*) P(\alpha,\alpha^*)$	

Application: Driven Harmonic Oscillator

$$\hat{H}_0 = \hbar\omega\left(a^+a + \tfrac{1}{2}\right)$$

$$\hat{H}_{DRIVE} = \left(\lambda a^+ + \lambda^* a\right) \quad \text{Driving term cf. H0}$$

Hence:

$$i\hbar\partial_t\rho = \hbar\omega[a^+a,\rho] + \lambda[a^+,\rho] + \lambda^*[a,\rho]$$

Can be converted to a differential equation:

$$* \boxed{\partial_t P(\alpha,\alpha^*) = -i\left(-\omega\frac{\partial}{\partial\alpha}\alpha + \omega\frac{\partial}{\partial\alpha^*}\alpha^*\right) - \frac{\lambda}{\hbar}\frac{\partial}{\partial\alpha} + \frac{\lambda^*}{\hbar}\frac{\partial}{\partial\alpha^*}\Big)P}$$

Solution:

$$P(\alpha,\alpha^*,t) = \partial^2(\alpha - \alpha(t))$$

$$\alpha(t) = \alpha(0)e^{i\omega t} + i\int_0^t dt'\, e^{-i(\omega(t-t'))}\lambda(t')/\hbar$$

* Solution: $\alpha = x + iy$

$$\frac{\partial}{\partial\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$